MIXED MULTIPLICITIES OF MULTI-GRADED ALGEBRAS OVER NOETHERIAN LOCAL RINGS *

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ABSTRACT: Let $S = \bigoplus_{n_1,\ldots,n_s \geq 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard multi-graded algebra over a Noetherian local ring A. This paper first expresses mixed multiplicities of S in term of Hilbert-Samuel multiplicity that explained the mixed multiplicities S as the Hilbert-Samuel multiplicities for quotient modules of $S_{(n_1,\ldots,n_s)}$. As an application, we get formulas for the mixed multiplicities of ideals that covers the main result of Trung-Verma in [TV].

1. Introduction

Throughout this paper, let $(A, \mathfrak{m})$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$; $S = \bigoplus_{n_1,\ldots,n_s \geq 0} S_{(n_1,\ldots,n_s)}$ $(s > 0)$ a finitely generated standard s-graded algebra over A. Let J be an $\mathfrak{m}$-primary ideal of A. Set

$$D_J(S) = \bigoplus_{n \geq 0} \frac{J^n S_{(n,\ldots,n)}}{J^{n+1} S_{(n,\ldots,n)}}$$

and $\ell = \dim D_{\mathfrak{m}}(S)$. Then

$$\ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big)$$

is a polynomial of total degree $\ell - 1$ in $n_0, n_1, \ldots, n_s$ for all large $n_0, n_1, \ldots, n_s$ (see Section 3). If we write the term of total degree $\ell - 1$ in this polynomial in the form

$$\sum_{k_0+k_1+\cdots+k_s=\ell-1} e(J, k_0, k_1, \ldots, k_s, S) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{n_0! n_1! \cdots n_s!}$$

then $e(J, k_0, k_1, \ldots, k_s, S)$ are non-negative integers not all zero and called the *mixed multiplicity of* S *of type* $(k_0, k_1, \ldots, k_s)$ *with respect to* J.

In particular, when $S = A[I_1t_1, \ldots, I_st_s]$ is a multi-graded Rees algebra of ideals $I_1, \ldots, I_s$, $e(J, k_0, k_1, \ldots, k_s, S)$ exactly is the mixed multiplicity of a set of ideals $(J, I_1, \ldots, I_s)$ in local ring A(see [Ve2] or [HHRT]).

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Mixed multiplicities of ideals were first introduced by Teissier and Risler in 1973 for two $\mathfrak{m}$-primary ideals and in this case they can be interpreted as the multiplicity of general elements [Te]. Next, Rees in 1984 proved that each mixed multiplicities of a set of $\mathfrak{m}$-primary ideals is the multiplicity of a joint reduction of them [Re]. In general, mixed multiplicities have been mentioned in the works of Verma, Katz, Swanson and other authors, see e.g. [Ve1], [Ve2], [Ve3], [Sw], [HHRT], [KV], [Tr1]. By using the concept of (FC)-sequences, Viet in 2000 showed that one can transmute mixed multiplicities of a set of arbitrary ideals into Hilbert-Samuel multiplicities [Vi]. Trung in 2001 gave the criteria for the positivity of mixed multiplicities of an ideal I [Tr2]. Similar to the methods of Viet [Vi], Trung and Verma in 2007 characterize also mixed multiplicities of a set of ideals, in term of superficial sequences [TV]. Moreover, some another authors have extended mixed multiplicities of a set of ideals to modules, e.g. Kirby and Rees in [KR1, KR2]. Kleiman and Thorup in [KT1, KT2], Manh and Viet in [MV1]. In a recent paper [VM] Viet and Manh investigated the mixed multiplicities of multigraded algebas over Artinian local rings.

In this paper, we consider mixed multiplicities of multi-graded algebra S over Noetherian local ring. Our aim is to characterize mixed multi-graded of S with respect to J in term of Hilbert-Samuel multiplicity (Theorem 3.3, Sect.3). As an application, we get a version of Theorem 3.3 for mixed multiplicities of arbitrary ideals in local rings (Theorem 4.3, Sect.4) that covers the main result in [TV].

The paper is divided in four sections. In Section 2, we investigate (FC)-sequences of multi-graded algebras. Section 3 gives some results on expressing mixed multiplicities of multi-graded algebras in terms of Hilbert-Samuel multiplicity. Section 4 devoted to the discussion of mixed multiplicities of arbitrary ideals in local rings.

2. (FC)-sequences of multi-graded algebras

The author in [Vi] built (FC)-sequences of ideals in local ring for calculating mixed multiplicities of set of ideals. In order to study mixed multiplicities of multi-graded algebras, this section introduces (FC)-sequences in multi-graded algebras and gives some important properties of these sequences.

Set $\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n=0}^{\infty}(\mathfrak{a} : \mathfrak{b}^n)$, and

$$\begin{aligned}
&(M : N)_A = \{a \in A \mid aN \subset M\};\\
&S_+ = \bigoplus_{n_1+\cdots+n_s>0} S_{(n_1,\ldots,n_s)};\\
&S_i = S_{(0,\ldots,\underset{(i)}{1},\ldots,0)};\\
&S_i^+ = S_iS = \bigoplus_{n_i>0} S_{(n_1,\ldots,n_s)} (i = 1, 2, \ldots, s);\\
&S_{++} = S_1^+ \cap \cdots \cap S_s^+ = \bigoplus_{n_1,\ldots,n_s>0} S_{(n_1,\ldots,n_s)} = S_{(1,\ldots,1)}S.
\end{aligned}$$

Definition 2.1. *Let $S = \bigoplus_{n_1,\dots,n_s \geq 0} S_{(n_1,\dots,n_s)}$ be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and let I be an ideal of A. A homogeneous element $x \in S$ is called a weak-(FC)-element of S with respect to I if there exists $i \in \{1, 2, \dots, s\}$ such that $x \in S_i$ and*

- (FC$_1$): $xS_{(n_1,\dots,n_i-1,\dots,n_s)} \cap I^{n_0} S_{(n_1,\dots,n_s)} = xI^{n_0} S_{(n_1,\dots,n_i-1,\dots,n_s)}$ *for all large* $n_0, n_1, \dots, n_s$.
- (FC$_2$): x *is a filter-regular element of* S*, i.e.,* $(0 : x)_{(n_1,\dots,n_s)} = 0$ *for all large* $n_1, \dots, n_s$.

Let $x_1, \dots, x_t$ be a sequence in S. We call that $x_1, \dots, x_t$ is a weak-(FC)-sequence of S with respect to I if $\bar{x}_{i+1}$ is a weak-(FC)-element of $S/(x_1, \dots, x_i)S$ with respect to I for all $i = 0, 1, \dots, t-1$, where $\bar{x}_{i+1}$ is the image of x_{i+1} in $S/(x_1, \dots, x_i)S$.

Example 2.2: Let $R = A[X_1, X_2, \dots, X_t]$ be the ring of polynomial in t indeterminates $X_1, X_2, \dots, X_t$ with coefficients in A ($\dim A = d > 0$). Then $R = \bigoplus_{m \geq 0} R_m$ is a finitely generated standard graded algebra over A, where R_m is the set of all homogeneous polynomials of degree m and the zero polynomial. It is well known that $X_1, X_2, \dots, X_t$ is a regular sequence of R. Let I be an ideal of A. It is easy to see that $X_1 R_{m-1} \cap IR_m$ and $IX_1 R_{m-1}$ are both the set of all homogeneous polynomials of degree m with coefficients in $\mathfrak{I}$ and divided by X_1. Hence $X_1 R_{m-1} \cap IR_m = IX_1 R_{m-1}$ for any ideal I of A. Using the results just obtained and the fact that

$$R/(X_1, \dots, X_i)R = A[X_{i+1}, \dots, X_t]$$

for all $i < t$, we immediately show that $X_1, X_2, \dots, X_t$ be a weak-(FC)-sequence of R with respect to I for any ideal I of A.

Now, we give some comments on weak-(FC)-sequences of a finitely generated standard multi-graded algebra over A by the following remark.

Remark 2.3.

(i) By Artin-Rees lemma, there exists integer $u_1, u_2, \dots, u_s$ such that

$$\begin{aligned}(0 : S_{++}^{\infty}) \bigcap S_{(n_1,\dots,n_s)} &= S_{(n_1-u_1,\dots,n_s-u_s)}((0 : S_{++}^{\infty}) \bigcap S_{(u_1,\dots,u_s)}) \\ &\subseteq S_{(n_1-u_1,\dots,n_s-u_s)}(0 : S_{++}^{\infty})\end{aligned}$$

for all $n_1 \geq u_1, \dots, n_s \geq u_s$. Since $S_{(n_1-u_1,\dots,n_s-u_s)}(0 : S_{++}^{\infty}) = 0$ for all large enough $n_1, \dots, n_s$, it follows that $(0 : S_{++}^{\infty})_{(n_1,\dots,n_s)} = (0 : S_{++}^{\infty}) \bigcap S_{(n_1,\dots,n_s)} = 0$ for all large enough $n_1, \dots, n_s$.

(ii) Let $x \in S$ be a homogeneous element. If $0 : x \subseteq 0 : S_{++}^{\infty}$ then, by (i),

$$(0 : x)_{(n_1,\dots,n_s)} \subseteq (0 : S_{++}^{\infty})_{(n_1,\dots,n_s)} = 0$$

for all large $n_1, \ldots, n_s$. Thus x is a filter-regular element of S. Conversely, suppose that x is a filter-regular element of S. We have

$$S_{(n_1,\ldots,n_s)}(0:x)_{(v_1,\ldots,v_s)} \subseteq (0:x)_{(n_1+v_1,\ldots,n_s+v_s)} = 0$$

for all large $n_1, \ldots, n_s$ and all $v_1, \ldots, v_s$. It implies that

$$(0:x)_{(v_1,\ldots,v_s)} \subseteq (0:S_{++}^n) \subseteq (0:S_{++}^\infty)$$

for all large n and all $v_1, \ldots, v_s$. Hence $(0:x) \subseteq (0:S_{++}^\infty)$. Therefore x is a filter-regular element of S if and only if $0:x \subseteq 0:S_{++}^\infty$.

(iii) Suppose that $x \in S_i$ is a filter-regular element of S. Consider

$$\lambda_x : S_{(n_1,\ldots,n_i,\ldots,n_s)} \longrightarrow xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}, y \mapsto xy.$$

It is clear that λ_x is surjective and $\ker\lambda_x = (0:x) \cap S_{(n_1,\ldots,n_s)} = 0$ for all large $n_1, \ldots, n_s$. Therefore, $S_{(n_1,\ldots,n_i,\ldots,n_s)} \cong xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. This follows that

$$IS_{(n_1,\ldots,n_i,\ldots,n_s)} \cong xIS_{(n_1,\ldots,n_i-1,\ldots,n_s)}$$

for all large $n_1, \ldots, n_s$ and for any ideal I of A.

(iv) If S_{++} is non-nilpotent then $S_{(n,\ldots,n)} \neq 0$ for all n. Hence, by Nakayama's Lemma, $(D_{\mathfrak{m}}(S))_n = \frac{\mathfrak{m}^n S_{(n,\ldots,n)}}{\mathfrak{m}^{n+1} S_{(n,\ldots,n)}} \neq 0$ for all n. It implies that $\dim D_{\mathfrak{m}}(S) \geq 1$.

The following lemma will play a crucial role for showing the existence of weak-(FC)-sequence.

Lemma 2.4 (Generalized Rees's Lemma). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$. Let $S = \bigoplus_{n_1,\ldots,n_s \geq 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard s-graded algebra over A; I be an ideal of A. Let Σ be a finite collection of prime ideals of S not containing $S_{(1,\ldots,1)}$. Then for each $i = 1, \ldots, s$, there exists an element $x_i \in S_i \backslash \mathfrak{m}S_i$, x_i not contained in any prime ideal in Σ, and a positive integer k_i such that*

$$x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} = x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$$

for all $n_i > k_i$ and all non-negative integers $n_0, n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$.

Proof. In the ring $S[t, t^{-1}]$ (t is an indeterminate), set

$$S^* = \bigoplus_{n_0 \in \mathbb{Z}} I^{n_0} S t^{n_0} = \bigoplus_{n_0 \in \mathbb{Z}; n_1,\ldots,n_s \geq 0} I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0}$$

where $I^n = A$ for $n \leq 0$. Then S^* is a Noetherian $(s+1)$-graded ring. From $u = t^{-1}$ is non-zero-divisor in S^*, by the Corollary of [Lemma 2.7, Re], the set of prime

associated with u^nS^* is independent on $n > 0$ and so is finite. We divide this set into two subsets: $\mathfrak{S}_1$ consisting of those containing S_i and $\mathfrak{S}_2$ those that do not (where $S_i = S_{(0,\ldots,\underset{(i)}{1},\ldots,0)} = S^*_{(0,0,\ldots,\underset{(i+1)}{1},\ldots,0)}$).

From $S_i/\mathfrak{m}S_i$ is a vector space over the infinite field k and the sets $\Sigma, \mathfrak{S}_2$ are both finite, we can choose $x_i \in S_i \backslash \mathfrak{m}S_i$ such that x_i is not contained in any prime ideal belonging to $\Sigma \cup \mathfrak{S}_2$. Set

$$M_n = \frac{(u^nS^* : x_i) \cap S^*}{u^nS^*}.$$

Then M_n is a S^*-module for any $n > 0$. We need must show that there exists a sufficiently large integer $N > 0$ such that $S_i^N M_n = 0$. Note that if $P \in \mathrm{Ass}_{S^*} M_n$ then $P \in \mathrm{Ass}_{S^*} S^*/u^nS^* = \mathfrak{S}_1 \cup \mathfrak{S}_2$, and there exists $z \in u^nS^* : x_i$ such that $P = u^nS^* : z$. Since $x_iz \in u^nS^*$, $x_i \in P$. So $P \in \mathfrak{S}_1$. Hence $S_i \subset P$. It follows that $S_i \subset \bigcap_{P \in \mathrm{Ass}_{S^*} M_n} P$. Therefore

$$S_i \subset \sqrt{\mathrm{Ann}_{S^*} M_n}.$$

Since S_i is finitely generated, there exists a sufficiently large integer $N > 0$ (how large depending on n) such that $S_i^N M_n = 0$. Hence, for all large $n_i > N$, any element of M_n of degree $(n_0, n_1, \ldots, n_s)$ is zero. This means that, for each $n > 0$, we have

$$(u^n I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0} : x_i) \bigcap S^* = u^n I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0} \tag{1}$$

for all large n_i and all non-negative integers $n_0, n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$.

Let $\mathfrak{b}$ denote the ideal of S^* consisting of all finite sums $\sum c_{n_0} t^{n_0}$ with

$$c_{n_0} \in x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)}.$$

Then $\mathfrak{b}$ has a finite generating set of the form $x_i b_i t^{n_0}$ with $b_i \in S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. Note that if $0 \neq a \in I^m S$ and $m \geq n_0$ then $at^{n_0} \in S^*$. Specially, if $n_0 < 0$ then $at^{n_0} \in S^*$ for all $a \in S$. Hence since the above generating set of $\mathfrak{b}$ is finite, it follows that there exists an integer q such that $u^q b_i t^{n_0} = b_i t^{n_0 - q} \in S^*$ for all element of this generating set (q is chosen such that $n_0 - q < 0$ for all n_0). Therefore $\mathfrak{b} \subseteq x_i S^* : u^q$.

Now, suppose that $z \in x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)}$. This means $zt^{n_0} \in \mathfrak{b}$. Because $\mathfrak{b} \subseteq x_i S^* : u^q$, $u^q z t^{n_0} = x_i w$, where $w \in S^*$. Since $z \in I^{n_0} S_{(n_1,\ldots,n_s)}$, it follows that $x_i w = u^q z t^{n_0} \in u^q I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0}$. Hence, by (1), we can find k_i such that

$$w \in (u^q I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0} : x_i) \bigcap S^* = u^q I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0}$$

for all $n_i > k_i$. Thus $u^q z t^{n_0} = x_i w \in x_i u^q I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0}$. Since u and t are non-zero-divisors in S^*, $z \in x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. Hence if $n_i > k_i$,

$$x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} \subset x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}.$$

Consequently, $x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} = x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. ■

The following proposition will show the existence of weak-(FC) sequence.

Proposition 2.5. *Suppose that S_{++} is non-nilpotent. Then for any $1 \le i \le s$, there exists a weak-(FC)-element $x \in S_i$ of S with respect to I.*

Proof. Since S_{++} is non-nilpotent, $S/0 : S_{++}^{\infty} \neq 0$. Set

$$\Sigma = \mathrm{Ass}_S(S/0 : S_{++}^{\infty}) = \{P \in \mathrm{Ass}S \mid P \not\supseteq S_{(1,\ldots,1)}\}.$$

Then Σ is finite. By Lemma 2.4, for each $i = 1, \ldots, s$, there exists $x \in S_i \setminus \mathfrak{m}S_i$ such that $x \notin P$ for all $P \in \Sigma$ and

$$xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)} = xI^{n_0}S_{(n_1,\ldots,n_s)}.$$

Thus x satisfies the condition (FC$_1$). Since $x \notin P$ for all $P \in \Sigma$, $0 : x \subset 0 : S_{++}^{\infty}$. Hence by Remark 2.3(ii), x satisfies the condition (FC$_2$). ■

3. Mixed multiplicities of multi-graded algebras

This section first determines mixed multiplicities of multi-graded algebras defined over a Noetherian local ring, next answers to the question when these mixed multiplicities are positive and characterizes them in term of Hilbert-Samuel multiplicities.

Let $S = \bigoplus_{n_1,\ldots,n_s \ge 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and J be an $\mathfrak{m}$-primary ideal of A. Since

$$\bigoplus_{n_0,n_1,\ldots,n_s \ge 0} \frac{J^{n_0}S_{(n_1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_s)}}$$

is a finitely generated standard s-graded algebra over Artinian local ring A/J, by [HHRT, Theorem 4.1],

$$\ell_A\Big(\frac{J^{n_0}S_{(n_1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_s)}}\Big)$$

is a polynomial for all large $n_0, n_1, \ldots, n_s$. Denote by $P(n_0, n_1, \ldots, n_s)$ this polynomial. Set

$$D_J(S) = \bigoplus_{n \ge 0} \frac{J^nS_{(n,\ldots,n)}}{J^{n+1}S_{(n,\ldots,n)}}$$

and $\ell = \dim D_{\mathfrak{m}}(S)$. By Remark 2.3(iv), $\ell \ge 1$. Note that $\dim D_J(S) = \dim D_{\mathfrak{m}}(S)$ for all $\mathfrak{m}$-primary ideal J of A and $\deg P(n_0, n_1, \ldots, n_s) = \deg P(n, n, \ldots, n)$. Since

$$P(n, n, \ldots, n) = \ell_A\Big(\frac{J^nS_{(n,\ldots,n)}}{J^{n+1}S_{(n,\ldots,n)}}\Big) = \ell_A(D_J(S)_n)$$

for all large n, it follows that $\deg P(n, n, \ldots, n) = \dim D_J(S) - 1 = \ell - 1$. Hence $\deg P(n_0, n_1, \ldots, n_s) = \ell - 1$.

If we write the term of total degree $\ell - 1$ of P in the form

$$\sum_{k_0+k_1+\cdots+k_s=\ell-1} e(J, k_0, k_1, \ldots, k_s, S)\frac{n_0^{k_0} n_1^{k_1}\cdots n_s^{k_s}}{n_0! n_1! \cdots n_s!}$$

then $e(J, k_0, k_1, \ldots, k_s, S)$ are non-negative integers not all zero and called the *mixed multiplicity of S of type $(k_0, k_1, \ldots, k_s)$ with respect to J.*

From now on, the notation $e_A(J, M)$ will mean the Hilbert-Samuel multiplicity of A-module M with respect to ideal $\mathfrak{m}$-primary J of A. We shall begin the section with the following lemma.

Lemma 3.1. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and J be an $\mathfrak{m}$-primary ideal of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Then $e(J, k_0, 0, \ldots, 0, S) \neq 0$ if and only if $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \ell$. In this case, $e(J, k_0, 0, \ldots, 0, S) = e_A(J, S_{(n,\ldots,n)})$ for all large n.*

Proof. Denote by $P(n_0, n_1, \ldots, n_s)$ the polynomial of

$$\ell_A\Big(\frac{J^{n_0}S_{(n_1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_s)}}\Big).$$

Then P is a polynomial of total degree $\ell - 1$. By taking $n_1 = n_2 = \cdots = n_s = u$, where u is a sufficiently large integer, we get

$$e(J, k_0, 0, \ldots, 0, S) = \lim_{n_0\to\infty} \frac{(\ell-1)! P(n_0, u, \ldots, u)}{n_0^{\ell-1}}.$$

Since $P(n_0, u, \ldots, u) = \ell_A(\frac{J^{n_0}S_{(u,\ldots,u)}}{J^{n_0+1}S_{(u,\ldots,u)}})$, it follows that

$$\deg P(n_0, u, \ldots, u) = \dim_A S_{(u,\ldots,u)} - 1$$

and $e(J, k_0, 0, \ldots, 0, S) \neq 0$ if and only if

$$\deg P(n_0, u, \ldots, u) = \dim_A S_{(u,\ldots,u)} - 1 = \ell - 1.$$

Since A is Noetherian, $(0 : S^{\infty}_{(1,\ldots,1)})_A = (0 : S^{n}_{(1,\ldots,1)})_A = (0 : S_{(n,\ldots,n)})_A$ for all large n. Hence if u is chosen sufficiently large, we have

$$\dim_A S_{(u,\ldots,u)} = \dim A/(0 : S_{(u,\ldots,u)})_A = \dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A.$$

Therefore $e(J, k_0, 0, \ldots, 0, S) \neq 0$ if and only if $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \ell$. Finally, if $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \ell$ then $\dim_A S_{(n,\ldots,n)} - 1 = \ell - 1$ for all large n and hence

$$\begin{aligned} e_A(J, S_{(n,\ldots,n)}) &= \lim_{n_0\to\infty} \frac{(\ell-1)!\ell_A(\frac{J^{n_0}S_{(n,\ldots,n)}}{J^{n_0+1}S_{(n,\ldots,n)}})}{n_0^{\ell-1}} \\ &= \lim_{n_0\to\infty} \frac{(\ell-1)! P(n_0, n, \ldots, n)}{n_0^{\ell-1}} = e(J, k_0, 0, \ldots, 0, S) \end{aligned}$$

for all large integer n. ■

Proposition 3.2. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and J be an $\mathfrak{m}$-primary ideal of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Assume that $e(J, k_0, k_1, \ldots, k_s, S) \neq 0$, where $k_0, k_1, \ldots, k_s$ are non-negative integers such that $k_0 + k_1 + \cdots + k_s = \ell - 1$. Then*

(i) *If $k_i > 0$ and $x \in S_i$ is a weak-(FC)-element of S with respect to J then*

$$e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/xS),$$

and $\dim D_{\mathfrak{m}}(S/xS) = \ell - 1$.

(ii) *There exists a weak-(FC)-sequence of $t = k_1 + \cdots + k_s$ elements of S in $\bigcup_{i=1}^s S_i$ with respect to J consisting of k_1 elements of S_1, …, k_s elements of S_s.*

Proof. The proof of (i): Denote by $P(n_0, n_1, \ldots, n_s)$ the polynomial of

$$\ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big).$$

Then $\deg P = \ell - 1$. Since x satisfies the condition (FC_1), for all large $n_0, n_1, \ldots, n_s$, we have

$$\begin{aligned}
&\ell_A\Big(\frac{J^{n_0}(S/xS)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\ldots,n_s)}}\Big) = \ell_A\Big(\frac{J^{n_0}(S_{(n_1,\ldots,n_s)}/xS_{(n_1,\ldots,n_i-1,\ldots,n_s)})}{J^{n_0+1}(S_{(n_1,\ldots,n_s)}/xS_{(n_1,\ldots,n_i-1,\ldots,n_s)})}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{(J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}) \cap J^{n_0} S_{(n_1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0} S_{(n_1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big) - \ell_A\Big(\frac{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big) - \ell_A\Big(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big)\\
&= \ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big) - \ell_A\Big(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0+1} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\Big).
\end{aligned}$$

Since x is a filter-regular element of S, it follows by Remark 2.3(iii) that

$$J^{n_0} S_{(n_1,\ldots,n_i,\ldots,n_s)} \cong xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$$

for all n_0 and all large $n_1, \ldots, n_s$. Thus we have an isomorphism of A-modules

$$\frac{xJ^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0+1}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}} \cong \frac{J^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}$$

for all large $n_0, n_1, \ldots, n_s$. So

$$\ell_A(\frac{xJ^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0+1}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}) = \ell_A(\frac{J^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}).$$

Hence

$$\ell_A(\frac{J^{n_0}(S/xS)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\ldots,n_s)}}) = \ell_A(\frac{J^{n_0}S_{(n_1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_s)}}) - \ell_A(\frac{J^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}})$$

for all large $n_0, n_1, \ldots, n_s$. Denote by $Q(n_0, n_1, \ldots, n_s)$ the polynomial of

$$\ell_A(\frac{J^{n_0}(S/xS)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\ldots,n_s)}}).$$

From the above fact, we get

$$Q(n_0, n_1, \ldots, n_s) = P(n_0, n_1, \ldots, n_i, \ldots, n_s) - P(n_0, n_1, \ldots, n_i - 1, \ldots, n_s).$$

Since $e(J, k_0, k_1, \ldots, k_s, S) \neq 0$ and $k_i > 0$, it implies that $\deg Q = \deg P - 1$ and

$$e(J, k_0, k_1, \ldots, k_i, \ldots, k_s, S) = e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/xS).$$

Note that $\deg Q = \dim D_{\mathfrak{m}}(S/xS) - 1$. Hence

$$\dim D_{\mathfrak{m}}(S/xS) = \deg Q + 1 = \deg P = \ell - 1.$$

The proof of (ii): The proof is by induction on $t = k_1 + \cdots + k_s$. For $t = 0$, the result is trivial. Assume that $t > 0$. Since $k_1 + \cdots + k_s = t > 0$, there exists $k_j > 0$. Since S_{++} is non-nilpotent, by Proposition 2.5, there exists a weak-(FC) element $x_1 \in S_j$ of S with respect to J. By (i),

$$e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/x_1S) = e(J, k_0, k_1, \ldots, k_s, S) \neq 0.$$

This follows that

$$\frac{J^{n_0}(S/x_1S)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/x_1S)_{(n_1,\ldots,n_s)}}$$

and so $(S/x_1S)_{(n_1,\ldots,n_s)} \neq 0$ for all large $n_1, \ldots, n_s$. Hence $(S/x_1S)_{++}$ is non-nilpotent. Since $k_1 + \cdots + k_j - 1 + \cdots + k_s = t - 1$, by the inductive assumption, there exists $t - 1$ elements $x_2, \ldots, x_t$ consisting of k_1 elements of S_1, ..., $k_j - 1$ elements of S_j, ..., k_s elements of S_s such that $\bar{x}_2, \ldots, \bar{x}_t$ is a weak-(FC)-sequence of S/x_1S with respect to J ($\bar{x}_i$ is initial form of x_i in S/x_1S, $i = 2, \ldots, t$). Hence $x_1, \ldots, x_t$ is a weak-(FC)-sequence of S with respect to J consisting of k_1 elements of S_1, ..., k_s elements of S_s. ■

The following theorem will give the criteria for the positivity of mixed multiplicities and characterize them in term of Hilbert-Samuel multiplicity.

Theorem 3.3. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent. Let J be an $\mathfrak{m}$-primary ideal of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Then the following statements hold.*

(i) *$e(J, k_0, k_1, \ldots, k_s, S) \neq 0$ if and only if there exists a weak-(FC)-sequence $x_1, \ldots, x_t$ $(t = k_1 + \cdots + k_s)$ of S with respect to J consisting of k_1 elements of S_1, ..., k_s elements of S_s and*

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) = \dim A/((x_1, \ldots, x_t)S : S^{\infty}_{(1,\ldots,1)})_A = \ell - t.$$

(ii) *Suppose that $e(J, k_0, k_1, \ldots, k_s, S) \neq 0$ and $x_1, \ldots, x_t$ $(t = k_1 + \cdots + k_s)$ is a weak-(FC)-sequence of S with respect to J consisting of k_1 elements of S_1, ..., k_s elements of S_s. Set $\bar{S} = S/(x_1, \ldots, x_t)S$. Then*

$$e(J, k_0, k_1, \ldots, k_s, S) = e_A(J, \bar{S}_{(n,\ldots,n)})$$

for all large n.

Proof. The proof of (i): First, we prove the necessary condition. By Proposition 3.2(ii), there exists a weak-(FC)-sequence $x_1, \ldots, x_t$ of S with respect to J consisting of k_1 elements of S_1, ..., k_s elements of S_s. Set $\bar{S} = S/(x_1, \ldots, x_t)S$. Applying Proposition 3.2(i) by induction on t, we get $\dim D_{\mathfrak{m}}(\bar{S}) = \ell - t$ and

$$0 \neq e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, 0, \ldots, 0, \bar{S}).$$

Hence by Lemma 3.1, $\dim A/(0 : \bar{S}^{\infty}_{(1,\ldots,1)})_A = \ell - t$. Since

$$\dim A/(0 : \bar{S}^{\infty}_{(1,\ldots,1)})_A = \dim A/((x_1, \ldots, x_t)S : S^{\infty}_{(1,\ldots,1)})_A,$$

it follows that $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) = \dim A/((x_1, \ldots, x_t)S : S^{\infty}_{(1,\ldots,1)})_A = \ell - t$. Now, we prove the sufficiently condition. Without loss of general, we may assume that $x_1 \in S_i$. Denote by $P(n_0, n_1, \ldots, n_s)$ and $Q(n_0, n_1, \ldots, n_s)$ the polynomials of

$$\ell_A\Big(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\Big) \ \text{ and } \ \ell_A\Big(\frac{J^{n_0}(S/x_1S)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/x_1S)_{(n_1,\ldots,n_s)}}\Big),$$

respectively. Then by the proof of Proposition 3.2(i) we have

$$Q(n_0, n_1, \ldots, n_s) = P(n_0, n_1, \ldots, n_i, \ldots, n_s) - P(n_0, n_1, \ldots, n_i - 1, \ldots, n_s).$$

This implies that $\deg Q \leq \deg P - 1$. Recall that $\deg Q = \dim D_{\mathfrak{m}}(S/x_1S) - 1$ and $\deg P = \dim D_{\mathfrak{m}}(S) - 1$. So $\dim D_{\mathfrak{m}}(S/x_1S) \leq \dim D_{\mathfrak{m}}(S) - 1$. Similarly, we have

$$\begin{aligned} \ell - t = &\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) \leq \dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_{t-1})S) - 1 \\ &\leq \cdots \leq \dim D_{\mathfrak{m}}(S/x_1S) - (t-1) \leq \dim D_{\mathfrak{m}}(S) - t = \ell - t. \end{aligned}$$

This fact follows $\dim D_{\mathfrak{m}}(S/x_1S) = \dim D_{\mathfrak{m}}(S) - 1$. Thus $\deg Q = \deg P - 1$. Hence

$$e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/x_1S).$$

By induction we have $e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, 0, \ldots, 0, \bar{S})$. Since

$$\dim A/(0 : \bar{S}^{\infty}_{(1,\ldots,1)})_A = \dim A/((x_1, \ldots, x_t)S : S^{\infty}_{(1,\ldots,1)})_A = \ell - t = \dim D_{\mathfrak{m}}(\bar{S}),$$

it follows, by Lemma 3.1, that $e(J, k_0, 0, \ldots, 0, \bar{S}) \neq 0$. Hence

$$e(J, k_0, k_1, \ldots, k_s, S) \neq 0.$$

The proof of (ii): Applying Proposition 3.2(i), by induction on t, we obtain

$$0 \neq e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, 0, \ldots, 0, \bar{S}).$$

On the other hand, by Lemma 3.1, $e(J, k_0, 0, \ldots, 0, \bar{S}) = e_A(J, \bar{S}_{(n,\ldots,n)})$ for all large integer n. Hence $e(J, k_0, k_1, \ldots, k_s, S) = e_A(J, \bar{S}_{(n,\ldots,n)})$ for all large n. ■

Remark 3.4. From the proof of Theorem 3.3 we get some comments as following.

(i) Assume that $x_1, \ldots, x_t$ is a weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J. If $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) = \dim D_{\mathfrak{m}}(S) - t$ then $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_i)S) = \dim D_{\mathfrak{m}}(S) - i$ for all $1 \leq i \leq t$.

(ii) If $k_i > 0$ and $x \in S_i$ is a weak-(FC)-sequence of S with respect to J such that $\dim D_{\mathfrak{m}}(S/xS) = \dim D_{\mathfrak{m}}(S) - 1$ then

$$e(J, k_0, k_1, \ldots, k_i, \ldots, k_s, S) = e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/xS).$$

(iii) If $e(J, k_0, k_1, \ldots, k_s, S) \neq 0$ then for every weak-(FC)-sequence $x_1, \ldots, x_t$ ($t = k_1 + \cdots + k_s$) of S with respect to J consisting of k_1 elements of S_1, ..., k_s elements of S_s we always have

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) = \dim A/((x_1, \ldots, x_t)S : S^{\infty}_{(1,\ldots,1)})_A = \ell - t.$$

(iv) Suppose that $x_1, \ldots, x_t$ is a weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J. Then $\dim D_{\mathfrak{m}}(S/x_1S) \leq \dim D_{\mathfrak{m}}(S) - 1$. By induction we have

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) \leq \dim D_{\mathfrak{m}}(S) - t = \ell - t.$$

If $\ell = t$ then $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)S) \leq 0$. Hence $(S/(x_1, \ldots, x_t)S)_{++}$ is nilpotent by Remark 2.3(iv). So $x_1, \ldots, x_t$ is a maximal weak-(FC)-sequence. This fact follows that the length of every weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J is not greater than ℓ.

Example 3.5: Let $R = A[X,Y]$ be a polynomial rings of indeterminates X,Y; $\dim A = d > 2$. Then R is a finitely generated standard 2-graded algebra over A with $\deg X = (1,0), \deg Y = (0,1)$ and

$$\dim D_{\mathfrak{m}}(R) = \dim\left[\oplus_{n\geq 0}\frac{\mathfrak{m}^n(XY)^n}{\mathfrak{m}^{n+1}(XY)^n}\right] = \dim\left(\oplus_{n\geq 0}\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}\right) = \dim A.$$

It can be verified that X,Y is a weak-(FC)-sequence of R with respect to $\mathfrak{m}$. Since $\dim D_{\mathfrak{m}}(R/(X)) = \dim(A/\mathfrak{m}) = 0$ and $d > 2$, $\dim D_{\mathfrak{m}}(R/(X)) < \dim D_{\mathfrak{m}}(R) - 1$.

From Theorem 3.3, in the case $s = 1$, we get the following result for a graded algebra $S = \bigoplus_{n\geq 0} S_n$.

Corollary 3.6. *Let $S = \bigoplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over A such that $S_+ = \bigoplus_{n>0} S_n$ is non-nilpotent and J be an $\mathfrak{m}$-primary ideal of A. Set $D_J(S) = \bigoplus_{n\geq 0} J^n S_n/J^{n+1}S_n$ and $\dim D_{\mathfrak{m}}(S) = \ell$. Suppose that $x_1, \ldots, x_q$ is a maximal weak-(FC)-sequence in S_1 of S with respect to J satisfying the condition $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_q)S) = \ell - q$. Then*

(i) *$e(J, \ell - i - 1, i, S) \neq 0$ if and only if $i \leq q$ and $\dim A/((x_1, \ldots, x_i)S : S_1^\infty)_A = \ell - i$.*

(ii) *If $e(J, \ell - i - 1, i, S) \neq 0$ then $e(J, \ell - i - 1, i, S) = e_A(J, S_n/(x_1, \ldots, x_i)S_{n-1})$ for all large n.*

Proof. By Theorem 3.3(ii) we immediately get (ii). Now let us to prove the part (i). The "if" part. Assume that $e(J, \ell - i - 1, i, S) \neq 0$. First, we show that $i \leq q$. Assume the contrary that $i > q$. Since $x_1, \ldots, x_q$ is a weak-(FC)-sequence in S_1 of S with respect to J, applying Proposition 3.2(i) by induction on q,

$$0 \neq e(J, \ell - i - 1, i, S) = e(J, \ell - i - 1, i - q, \bar{S}),$$

where $\bar{S} = S/(x_1, \ldots, x_q)S$. Since $e(J, \ell - i - 1, i - q, \bar{S}) \neq 0$ and $i - q > 0$, by Proposition 3.2(ii), there exists an element $x \in S_1$ such that $\bar{x}$ (the initial form of x in $\bar{S}$) is a weak-(FC)-element of $\bar{S}$ with respect to J. By Proposition 3.2(i), $\dim D_{\mathfrak{m}}(\bar{S}/x\bar{S}) = \ell - q - 1$. Hence $x_1, \ldots, x_q, x$ is a weak-(FC)-sequence in S_1 of S with respect to J satisfying the condition

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_q, x)S) = \ell - q - 1.$$

We thus arrive at a contradiction. Hence $i \leq q$. Since $e(J, \ell - i - 1, i, S) \neq 0$, by Remark 3.4(iii), $\dim A/((x_1, \ldots, x_i)S : S_1^\infty)_A = \ell - i$. We turn to the proof of sufficiency. Suppose that $i \leq q$ and

$$\dim A/((x_1, \ldots, x_i)S : S_1^\infty)_A = \ell - i.$$

Since $\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_q)S) = \ell - q$, it follows, by Remark 3.4(i), that

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_i)S) = \ell - i$$

for all $i \leq q$. Since $x_1, \ldots, x_i$ is a weak-(FC)-sequence of S with respect to J satisfying the condition

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_i)S) = \dim A/((x_1, \ldots, x_i)S : S_1^{\infty})_A = \ell - i$$

by Theorem 3.3(i), $e(J, \ell - i - 1, i, S) \neq 0$. ■

Example 3.7: Let $R = A[X_1, X_2, \ldots, X_t]$ be the ring of polynomial in t indeterminates $X_1, X_2, \ldots, X_t$ with coefficients in A ($\dim A = d > 0$). Then $R = \bigoplus_{m \geq 0} R_m$ is a finitely generated standard graded algebra over A (see Example 2.2). Let J is an $\mathfrak{m}$-primary ideal of A. By Example 2.2, $X_1, \ldots, X_t \in R_1$ is a weak-(FC)-sequence of R with respect to J. Denote by $P(n, m)$ the polynomial of $\ell_A(\frac{J^n R_m}{J^{n+1} R_m})$. We have

$$D_{\mathfrak{m}}(R) = \bigoplus_{T \geq 0} \frac{\mathfrak{m}^T R_T}{\mathfrak{m}^{T+1} R_T} = \frac{A[\mathfrak{m}X_1, \ldots, \mathfrak{m}X_t]}{\mathfrak{m}A[\mathfrak{m}X_1, \ldots, \mathfrak{m}X_t]}.$$

Since $\mathrm{ht}\mathfrak{m} > 0$, $\dim D_{\mathfrak{m}}(R) = \dim A + t - 1 = d + t - 1$. Hence $\deg P(n, m) = d + t - 2$. It is clear that $R/(X_1, \ldots, X_i)R = A[X_{i+1}, \ldots, X_t]$ for all $i \leq t$. Hence

$$\dim D_{\mathfrak{m}}(R/(X_1, \ldots, X_i)R) = \dim D_{\mathfrak{m}}(R) - i$$

for all $i \leq t$. Let us calculate $e(J, k_0, k_1, R)$, with $k_0 + k_1 = d + t - 1$. First, we consider the case $k_1 \geq t$. Since $X_1, \ldots, X_{t-1}$ is a weak-(FC)-sequence of R with respect to J and $\dim D_{\mathfrak{m}}(R/(X_1, \ldots, X_i)R) = \dim D_{\mathfrak{m}}(R) - i$ for all $i \leq t - 1$, by Remark 3.4(ii),

$$e(J, k_0, k_1, R) = e(J, k_0, k_1 - (t-1), R/(X_1, \ldots, X_{t-1})R) = e(J, k_0, k_1 - t + 1, A[X_t]).$$

Denote by $Q(m, n)$ the polynomial of $\ell_A(\frac{J^n X_t^m A}{J^{n+1} X_t^m A})$. Since X_t is regular element, $J^n X_t^m A \cong J^n A$. Thus, for all large n, m,

$$Q(n, m) = \ell_A(\frac{J^n X_t^m A}{J^{n+1} X_t^m A}) = \ell_A(\frac{J^n A}{J^{n+1} A}).$$

Hence $Q(n, m)$ is independent on m. Note that $e(J, k_0, k_1 - t + 1, A[X_t])$ is the coefficient of $\frac{1}{k_0!(k_1 - t + 1)!} n^{k_0} m^{k_1 - t + 1}$ in $Q(n, m)$. Since $k_1 - t + 1 > 0$, it follows that

$$e(J, k_0, k_1, R) = e(J, k_0, k_1 - t + 1, A[X_t]) = 0.$$

In the case $k_1 < t$, since $\dim D_{\mathfrak{m}}(R/(X_1, \ldots, X_{k_1})R) = \dim D_{\mathfrak{m}}(R) - k_1$, by Corollary 3.6(i), $e(J, k_0, k_1, R) \neq 0$ if and only if

$$\dim A/((X_1, \ldots, X_{k_1})R : R_1^{\infty})_A = d + t - 1 - k_1.$$

Since $X_1, \ldots, X_t$ are independent indeterminates,

$$((X_1, \ldots, X_{k_1})R : R_1^{\infty})_A \subset ((X_1, \ldots, X_{k_1})R : ((X_{k_1+1}, \ldots, X_t)A)^{\infty})_A = 0.$$

Hence $\dim A/((X_1, \ldots, X_{k_1})R : R_1^\infty)_A = \dim A = d$. Therefore, $e(J, k_0, k_1, R) \neq 0$ if and only if $k_1 = t-1$. For $k_1 = t-1$ (then $k_0 = d-1$), by Corollary 3.6(ii), we have

$$e(J, d-1, t-1, R) = e_A(J, R_u/(X_1, \ldots, X_{t-1})R_{u-1})$$

for all large u. Note that $R_u = (X_1, \ldots, X_t)^u A$. So $R_u/(X_1, \ldots, X_{t-1})R_{u-1} = X_t^u A$. Thus $e(J, R_u/(X_1, \ldots, X_{t-1})R_{u-1}) = e_A(J, X_t^u A)$. Since X_t^u is regular element in $A[X_t]$, $X_t^u A \cong A$. Hence $e(J, d-1, t-1, R) = e_A(J, X_t^u A) = e_A(J, A)$. From the above facts we get

$$e(J, k_0, k_1, R) = \begin{cases} 0 & \text{if } k_1 \neq t-1 \\ e_A(J, A) & \text{if } k_1 = t-1 \end{cases}.$$

Therefore

$$P(n, m) = \frac{e(J, A)}{(d-1)!(t-1)!} n^{d-1} m^{t-1} + \{\text{terms of lower degree}\}.$$

Remark 3.8. Example 3.5 and Example 3.7 showed that for any weak-(FC)-sequence $x_1, \ldots, x_t$ of S with respect to J, one can get

$$\dim D_{\mathfrak{m}}(S/(x_1, \ldots, x_t)) < \dim D_{\mathfrak{m}}(S) - t,$$

and $\dim A/((x_1, \ldots, x_t)S : S_1^\infty)_A = \dim[A/(0 : S_1^\infty)_A] - t$ although

$$\dim A/((x_1, \ldots, x_i)S : S_1^\infty)_A \neq \dim[A/(0 : S_1^\infty)_A] - i$$

for some $i < t$. That is a difference of weak-(FC)-sequences in graded algebras and weak-(FC)-sequences of ideals in local rings.

4. Applications

As an application of Theorem 3.3, this section devoted to the discussion of mixed multiplicities of arbitrary ideals in local rings.

Throughout this section, let $(A, \mathfrak{m})$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$, and an ideal $\mathfrak{m}$-primary J, and $I_1, \ldots, I_s$ ideals of A such that $I = I_1 \cdots I_s$ is non-nilpotent. Set $S = A[I_1t_1, \ldots, I_st_s]$. Then

$$D_J(S) = \bigoplus_{n \geq 0} \frac{(JI)^n}{J(JI)^n} \text{ and}$$

$$\ell_A\Big(\frac{J^{n_0} S_{(n_1, \ldots, n_s)}}{J^{n_0+1} S_{(n_1, \ldots, n_s)}}\Big) = \ell_A\Big(\frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s}}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s}}\Big)$$

is a polynomial of total degree $\dim D_J(S) - 1$ for all large $n_0, n_1, \ldots, n_s$. By Proposition 3.1 in [Vi], the degree of this polynomial is $\dim A/0 : I^\infty - 1$. Hence

$\dim D_J(S) = \dim A/0 : I^\infty$. Set $\dim A/0 : I^\infty = \ell$. In this case, $e(J, k_0, k_1, \ldots, k_s, S)$ for $k_0 + k_1 + \cdots + k_s = \ell - 1$ is called the mixed multiplicity of ideals $(J, I_1, \ldots, I_s)$ of type $(k_0, k_1, \ldots, k_s)$ and one put

$$e(J, k_0, k_1, \ldots, k_s, S) = e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A)$$

(see [Ve2] or [HHRT]). By using the concept of (FC)-sequences of ideals, one transmuted mixed multiplicities of a set of arbitrary ideals into Hilbert-Samuel multiplicities [Vi].

Definition 4.1 [see Definition, Vi]**.** *Let $I_1, \ldots, I_s$ be ideals such that $I = I_1 \cdots I_s$ is a non nilpotent ideal. A element $x \in A$ is called an (FC)-element of A with respect to $(I_1, \ldots, I_s)$ if there exists $i \in \{1, 2, \ldots, s\}$ such that $x \in I_i$ and*

(FC$_1$): $(x) \cap I_1^{n_1} \cdots I_i^{n_i} \cdots I_s^{n_s} = xI_1^{n_1} \cdots I_i^{n_i-1} \cdots I_s^{n_s}$ *for all large $n_1, \ldots, n_s$.*

(FC$_2$): *x is a filter-regular element with respect to I, i.e., $0 : x \subseteq 0 : I^\infty$.*

(FC$_3$): $\dim A/[(x) : I^\infty] = \dim A/0 : I^\infty - 1$.

We call x a weak-(FC)-element with respect to $(I_1, \ldots, I_s)$ if x satisfies conditions (FC_1) and (FC_2).

Let $x_1, \ldots, x_t$ be a sequence in A. For each $i = 0, 1, \ldots, t-1$, set $A_i = A/(x_1, \ldots, x_i)S$, $\bar{I}_j = I_j[A/(x_1, \ldots, x_i)]$, $\bar{x}_{i+1}$ the image of x_{i+1} in A_i. Then

$x_1, \ldots, x_t$ is called a weak-(FC)-sequence of A with respect to $(I_1, \ldots, I_s)$ if $\bar{x}_{i+1}$ is a weak-(FC)-element of A_i with respect to $(\bar{I}_1, \ldots, \bar{I}_s)$ for all $i = 0, 1, \ldots, t-1$.

$x_1, \ldots, x_t$ is called an (FC)-sequence of A with respect to $(I_1, \ldots, I_s)$ if $\bar{x}_{i+1}$ is an (FC)-element of A_i with respect to $(\bar{I}_1, \ldots, \bar{I}_s)$ for all $i = 0, 1, \ldots, t-1$.

A weak-(FC)-sequence $x_1, \ldots, x_t$ is called a maximal weak-(FC)-sequence if IA_{t-1} is a non-nilpotent ideal of A_{t-1} and IA_t is a nilpotent ideal of A_t.

Remark 4.2.

(i) The condition (FC$_1$) in Definition 4.1 is a weaker condition than the condition (FC$_1$) of definition of (FC)-element in [Vi].

(ii) If $x \in I_i$ is a weak-(FC)-element with respect to $(J, I_1, \ldots, I_s)$, then it can be verified that x is also a weak-(FC)-element of S with respect to J as in Definition 2.1.

(iii) If $x_1, \ldots, x_t$ is an (FC)-sequence with respect to $(J, I_1, \ldots, I_s)$, then from the condition (FC$_3$) we follow that $\dim A/((x_1, \ldots, x_t)S : S^\infty_{(1,\ldots,1)})_A = \ell - t$. Hence

$$\dim D_J(S/(x_1, \ldots, x_t)S) = \dim A/((x_1, \ldots, x_t)S : S^\infty_{(1,\ldots,1)})_A = \ell - t$$

that as in the state of Theorem 3.3(i).

(iv) By Lemma 3.1, $e(J, k_0, 0, \ldots, 0, S) \neq 0$ if and only if $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \ell$. In this case, $e(J, k_0, 0, \ldots, 0, S) = e_A(J, S_{(n,\ldots,n)})$ for all large n. But since $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \dim A/0 : I^{\infty}$, $\dim A/(0 : S^{\infty}_{(1,\ldots,1)})_A = \ell$. Hence $e(J, k_0, 0, \ldots, 0, S) = e_A(J, S_{(n,\ldots,n)})$ for all large n. It is a plain fact that $e_A(J, S_{(n,\ldots,n)}) = e_A(J, I^n)$. On the other hand by the proof of Lemma 3.2 [Vi], $e_A(J, I^n) = e_A(J, A/0 : I^{\infty})$ for all large n. Hence $e(J, k_0, 0, \ldots, 0, S) = e_A(J, A/0 : I^{\infty})$.

Then as an immediate consequence of Theorem 3.3, we obtained a more favorite result than [Theorem 3.4, Vi](see Remark 4.2 (i)) as follows.

Theorem 4.3 [see Theorem 3.4, Vi]. *Let $(A, \mathfrak{m})$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$, and an ideal $\mathfrak{m}$-primary J, and $I_1, \ldots, I_s$ ideals of A such that $I = I_1 \cdots I_s$ is non nilpotent. Then the following statements hold.*

(i) *$e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) \neq 0$ if and only if there exists a weak-(FC)-sequence $x_1, \ldots, x_t$ with respect to $(J, I_1, \ldots, I_s)$ consisting of k_1 elements of I_1, ..., k_s elements of I_s and $\dim A/(x_1, \ldots, x_t) : I^{\infty} = \dim A/0 : I^{\infty} - t$.*

(ii) *Suppose that $e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) \neq 0$ and $x_1, \ldots, x_t$ is a weak-(FC)-sequence with respect to $(J, I_1, \ldots, I_s)$ consisting of k_1 elements of I_1, ..., k_s elements of I_s. Set $\bar{A} = A/(x_1, \ldots, x_t) : I^{\infty}$. Then*

$$e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) = e_A(J, \bar{A}).$$

Note that one can get this result by a minor improvement in the proof of [Proposition 3.3, Vi](see [DV1]). Moreover, the filtration version of Theorem 4.3 is proved in [DV2].

Recently, [DV1] and [DV2] showed that from Theorem 4.3 one rediscover the earlier result of Trung and Verma [TV] on mixed multiplicities of ideals. This fact proved that Theorem 3.3 covers the main results in [Vi] and [TV].

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